

ON A PERTURBATION DETERMINANT FOR ACCUMULATIVE OPERATORS

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ABSTRACT. For a purely imaginary sign-definite perturbation of a self-adjoint operator, we obtain exponential representations for the perturbation determinant in both upper and lower half-planes and derive respective trace formulas.

1. INTRODUCTION

The main goal of this note is to obtain new exponential representations for the perturbation determinant associated with a purely imaginary sign-definite perturbation of a self-adjoint operator.

Let $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \gtrless 0\}$. The starting point of our consideration is the exponential representation (see [21, Lemma 6.5]),

$$\det_{H/H_0}(z) = \exp \left(\frac{1}{\pi i} \int \frac{\zeta(\lambda)}{\lambda - z} d\lambda \right), \quad z \in \mathbb{C}_+, \quad (1.1)$$

for the perturbation determinant $\det_{H/H_0}(z) = \det((H - z)(H_0 - z)^{-1})$ associated with a self-adjoint operator H_0 and an accumulative operator $H = H_0 - iV$, where $V \geq 0$ is an element of the trace class S^1 . Here the nonnegative function $\zeta \in L^1(\mathbb{R})$ is given by

$$\zeta(\lambda) = \lim_{\varepsilon \rightarrow 0^+} \log |\det_{H/H_0}(\lambda + i\varepsilon)| \quad \text{for a.e. } \lambda \in \mathbb{R}.$$

(See also Theorem 6.6 as well as Lemma 5.6 and Theorem 5.7 in [21] for general additive and some singular non-additive perturbation results, respectively.) In Theorem 3.2, we give a new proof of (1.1) and, in Theorem 3.4, we obtain a complementary exponential representation for $\det_{H/H_0}(z)$ in \mathbb{C}_- .

Introducing the spectral shift function $\xi(\lambda)$ via the boundary value of the argument of the perturbation determinant

$$\xi(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0^+} \arg(\det_{H/H_0}(\lambda + i\varepsilon)) \quad \text{for a.e. } \lambda \in \mathbb{R},$$

we show (see Theorem 6.2) that ξ is *never* integrable whenever $H \neq H^*$; in fact, ξ is not even in $L^1_{w,0}(\mathbb{R})$ (see (4.4) for the definition of the weak zero space), but instead $\xi \in L^1_{w,0}(\mathbb{R}; \frac{d\lambda}{1+\lambda^2})$. By switching from Lebesgue integration to integration of type (A), we reconstruct the perturbation determinant from ξ in \mathbb{C}_+ (with formula mimicking the self-adjoint case) in Theorem 4.2.

Using the aforementioned representations for the perturbation determinant, in Theorem 5.1, we obtain a trace formula for rational functions vanishing at infinity with poles in both \mathbb{C}_- and \mathbb{C}_+ , which is an analog of a trace formula for contractions derived in [27–29]. Our

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approach to accumulative operators is based on complex and harmonic analysis, but, as distinct from [27–29], does not appeal to functional models of accumulative operators.

For the history (up to 1990) of perturbation determinants and associated trace formulas in the non-self-adjoint setting we refer to [4] where contributions to the field by H. Langer [20], L. A. Sahnovič [31], R. V. Akopjan [2, 3], P. Jonas [14, 15], V. A. Adamjan and B. S. Pavlov [5], A. V. Rybkin [26–28], M. G. Krein [19], H. Neidhardt [22, 23] are discussed in detail; for recent developments see [10, 21, 29, 30]. References to partial results for accumulative operators are also given in Remark 5.2. We also remark that various concepts of generalized integration, including the Kolmogorov-Titchmarsh A -integral, appeared to be rather useful in harmonic analysis [36], probability theory [16], as well as in perturbation theory for non-self-adjoint operators. In particular, the concept of A -integral has been systematically used for trace formulas associated with contractive trace class and special cases of Hilbert–Schmidt perturbations of a unitary operator (see [27–30] and the references therein).

We recall that a closed densely defined operator A is called accumulative if $\operatorname{Im} \langle Ah, h \rangle \leq 0$ for every h in the domain of A . Throughout the paper, we assume that every accumulative operator is maximal, which guarantees that \mathbb{C}_+ is a subset of the resolvent set of A [19, 24].

2. HERGLOTZ AND OUTER FUNCTIONS

We recall the canonical inner-outer factorization theorem for the Hardy classes H_p , $0 < p \leq \infty$, in the upper half-plane [17, Chapter VI:C, p. 119].

Theorem 2.1. *If $0 \not\equiv F \in H_p(\mathbb{C}_+)$, $0 < p \leq \infty$, then*

$$F(z) = I_F(z) \cdot O_F(z), \quad z \in \mathbb{C}_+,$$

where

(1) I_F is the inner factor of F given by

$$I_F(z) = e^{i\gamma + i\alpha z} B(z) \exp \left(\frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu_{\text{sing}}(\lambda) \right),$$

with

- (a) $\gamma \in \mathbb{R}$, $\alpha \geq 0$,
- (b) a Blaschke product

$$B(z) = \prod_{k=1}^{\infty} \left(e^{i\alpha_k} \frac{z - z_k}{z - \bar{z}_k} \right), \quad (2.1)$$

where z_k are the zeros of $F(z)$ in \mathbb{C}_+ and $\alpha_k \in \mathbb{R}$ are chosen so that $e^{i\alpha_k} \frac{i - z_k}{i - \bar{z}_k} \geq 0$,

- (c) $\mu_{\text{sing}} \geq 0$ a singular measure on \mathbb{R} satisfying $\int_{\mathbb{R}} \frac{d\mu_{\text{sing}}(\lambda)}{1 + \lambda^2} < \infty$,

(2) O_F is the outer factor of F given by

$$O_F(z) = \exp \left(\frac{1}{\pi i} \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) \log |F(\lambda + i0)| d\lambda \right). \quad (2.2)$$

Remark 2.2. (i) We have the Blaschke condition [17, Chapter VI:C]

$$\sum_{k=1}^{\infty} \frac{\operatorname{Im}(z_k)}{|z - z_k|^2} < \infty, \quad z \in \mathbb{C} \setminus (\mathbb{R} \cup \{z_k\}_{k=1}^{\infty}). \quad (2.3)$$

(ii) If, in addition, $|F(z)| \leq 1$, for $z \in \mathbb{C}_+$, then F can be factorized as

$$F(z) = B(z) \exp \left(\frac{i}{\pi} M(z) \right),$$

where $B(z)$ is the Blaschke product (2.1) and $M(z)$ is the Herglotz function

$$M(z) = \pi \alpha z + \pi \gamma + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu(\lambda), \quad \mu \geq 0.$$

(See, e.g., [1, Chapter VI, Section 59, Theorem 2] for representations of Herglotz functions.)

Definition 2.3. We say that F is an outer function if F is analytic on \mathbb{C}_+ , $|F|$ has finite boundary values a.e. on \mathbb{R} , $\log |F| \in L^1(\mathbb{R}; \frac{d\lambda}{1+\lambda^2})$, and for some $\theta \in \mathbb{R}$,

$$F(z) = e^{i\theta} O_F(z), \quad z \in \mathbb{C}_+,$$

where $O_F(z)$ is given by (2.2).

Theorem 2.4. If $M(z)$ is a Herglotz function, then the function $1 - iM(z)$ is outer in \mathbb{C}_+ .

Proof. The H^∞ -function $F(z) = (1 - iM(z))^{-1}$ has non-negative real part in the upper half-plane \mathbb{C}_+ . By [11, Corollary 4.8 (a)], $F(z)$ is an outer function, so is $F^{-1}(z) = 1 - iM(z)$. \square

Second proof. Since the function $(1 - iM(z))^{-1}$ is an analytic contractive function with no zeros in the upper half-plane by Remark 2.2, we have the representation

$$(1 - iM(z))^{-1} = \exp \left(\frac{i}{\pi} N(z) \right),$$

where $N(z)$ is a Herglotz function. Next, the function $i(1 - iM(z))$ is also Herglotz. Therefore, by the Aronszajn-Donoghue exponential Herglotz representation theorem (see, e.g., [12, Theorem 2.4]),

$$i(1 - iM(z)) = \exp \left(\frac{1}{\pi} L(z) \right)$$

for some absolutely continuously represented Herglotz function $L(z)$ without the linear term. Hence,

$$N(z) - iL(z) = 2k\pi^2 + \frac{\pi^2}{2}, \quad \text{for some } k \in \mathbb{Z}.$$

Since L has no linear term, so does N . Applying a variant of the brothers Riesz's theorem for the upper half-plane that states that if a complex-valued finite Borel measure μ on \mathbb{R} satisfies

$$\int_{\mathbb{R}} \frac{1 + z\lambda}{\lambda - z} d\mu(\lambda) = Az + B,$$

for all $z \in \mathbb{C}_+$ and some $A, B \in \mathbb{C}$, then $A = 0$ and μ is absolutely continuous, yields the representation

$$N(z) = \gamma + \int \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\nu(\lambda),$$

for some $\gamma \in \mathbb{R}$ and some absolutely continuous measure ν such that

$$\int_{\mathbb{R}} \frac{d\nu(\lambda)}{1 + \lambda^2} < \infty.$$

Now, the exponential representation

$$(1 - iM(z)) = \exp \left(-\frac{i}{\pi} N(z) \right)$$

shows that $1 - iM(z)$ is an outer function. \square

3. EXPONENTIAL REPRESENTATION FOR THE PERTURBATION DETERMINANT.

The main goal of this section is to obtain representations for the perturbation determinant associated with an accumulative trace class perturbation of a self-adjoint operator. As distinct from perturbation theory for self-adjoint operators, initial exponential representations for the perturbation determinant appear to be quite different in \mathbb{C}_- and \mathbb{C}_+ .

We start with the case of the upper half-plane and show that the perturbation determinant is an outer function in \mathbb{C}_+ by reducing the general case to the case of rank-one perturbations and obtain an exponential representation for it.

Lemma 3.1. *Let H_0 be a maximal accumulative operator, $\alpha > 0$, P a one-dimensional orthogonal projection, and let $H = H_0 - i\alpha P$. Then, the perturbation determinant $\det_{H/H_0}(z)$ is an outer function in the upper half-plane. Moreover,*

$$\det_{H/H_0}(z) = \exp \left(\frac{1}{\pi i} \int \frac{\zeta(\lambda)}{\lambda - z} d\lambda \right), \quad z \in \mathbb{C}_+,$$

where $\zeta \in L^1(\mathbb{R})$ is given by

$$\zeta(\lambda) = \lim_{\varepsilon \rightarrow 0^+} \log |\det_{H/H_0}(\lambda + i\varepsilon)| \geq 0, \quad \text{a.e. } \lambda \in \mathbb{R}, \quad (3.1)$$

and

$$\|\zeta\|_{L^1} = \pi\alpha. \quad (3.2)$$

Proof. Suppose that for every $g \in \mathcal{H}$,

$$Pg = \langle g, f \rangle f, \quad \|f\| = 1.$$

Then for all $z \in \rho(H_0)$,

$$\det_{H/H_0}(z) = \det(I - i\alpha P(H_0 - z)^{-1}) = 1 - i\alpha \langle (H_0 - z)^{-1} f, f \rangle. \quad (3.3)$$

Since $\alpha > 0$ and H_0 is an accumulative operator, the quadratic form $\alpha \langle (H_0 - z)^{-1} f, f \rangle$ is a Herglotz function in the upper half plane. Indeed, denote by \mathcal{L} the minimal self-adjoint dilation of the accumulative operator H_0 in a Hilbert space \mathcal{K} , $\mathcal{H} \subset \mathcal{K}$ (see [35] for details), so that

$$(H_0 - z)^{-1} = P_{\mathcal{H}}(\mathcal{L} - z)^{-1}|_{\mathcal{H}}, \quad z \in \mathbb{C}_+.$$

Hence, $\langle (H_0 - z)^{-1} f, f \rangle = \langle (\mathcal{L} - z)^{-1} \tilde{f}, \tilde{f} \rangle$ is a Herglotz function. Here $\tilde{f} = \mathcal{J}f$, with $\mathcal{J} : \mathcal{H} \rightarrow \mathcal{K}$ the natural imbedding of the Hilbert space \mathcal{H} into the Hilbert space \mathcal{K} .

By Theorem 2.4 and the representation (3.3), the perturbation determinant $\det_{H/H_0}(z)$ is an outer function in the upper half-plane. Therefore

$$\det_{H/H_0}(z) = e^{i\gamma} \exp \left(\frac{1}{\pi i} \int \zeta(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\lambda \right), \quad z \in \mathbb{C}_+,$$

where $\gamma \in \mathbb{R}$, the function $\zeta(\lambda)$ is given by (3.1), and

$$\int \frac{|\zeta(\lambda)|}{1 + \lambda^2} d\lambda < \infty.$$

Since

$$\operatorname{Re}(\det_{H/H_0}(z)) = 1 - \operatorname{Re}(i\alpha \langle (H_0 - z)^{-1}f, f \rangle) \geq 1, \quad z \in \mathbb{C}_+,$$

and hence,

$$|\det_{H/H_0}(z)| \geq 1, \quad z \in \mathbb{C}_+,$$

the function ζ given by (3.1) is non-negative almost everywhere. We also have the representation

$$\det_{H/H_0}(z) = 1 - i\alpha \langle (\mathcal{L} - z)^{-1}\tilde{f}, \tilde{f} \rangle, \quad z \in \mathbb{C}_+.$$

Since $\langle E_{\mathcal{L}}(\cdot)\tilde{f}, \tilde{f} \rangle$ is a finite measure, where $E_{\mathcal{L}}$ is the spectral measure of \mathcal{L} , we have the asymptotics

$$\det_{H/H_0}(iy) = 1 + \frac{\alpha}{y} + o(y^{-1}) \quad \text{as } y \rightarrow +\infty. \quad (3.4)$$

Hence,

$$\sup_{y>0} \left| \frac{y}{\pi i} \int \zeta(\lambda) \left(\frac{1}{\lambda - iy} - \frac{\lambda}{1 + \lambda^2} \right) d\lambda \right| < \infty,$$

which proves (see, e.g., [1, Chapter VI, Section 59, Theorem 3]) that ζ is an integrable function and, therefore, the perturbation determinant admits the representation

$$\det_{H/H_0}(z) = \exp \left(\frac{1}{\pi i} \int \frac{\zeta(\lambda)}{\lambda - z} d\lambda \right), \quad z \in \mathbb{C}_+.$$

One then observes that

$$\det_{H/H_0}(iy) = \exp \left(\frac{1}{\pi y} \int_{\mathbb{R}} \zeta(\lambda) d\lambda + o(y^{-1}) \right) \quad \text{as } y \rightarrow +\infty. \quad (3.5)$$

Comparing (3.4) and (3.5) yields

$$\int_{\mathbb{R}} \zeta(\lambda) d\lambda = \pi\alpha,$$

which proves (3.2), since ζ is non-negative a.e. \square

Theorem 3.2. *Let H_0 be a maximal accumulative operator, $0 \leq V = V^* \in S^1$, and let $H = H_0 - iV$. Then, the perturbation determinant $\det_{H/H_0}(z)$ is an outer function in \mathbb{C}_+ . Moreover,*

$$\det_{H/H_0}(z) = \exp \left(\frac{1}{\pi i} \int \frac{\zeta(\lambda)}{\lambda - z} d\lambda \right), \quad z \in \mathbb{C}_+, \quad (3.6)$$

where

$$\zeta(\lambda) = \lim_{\varepsilon \rightarrow 0^+} \log |\det_{H/H_0}(\lambda + i\varepsilon)| \geq 0 \quad \text{a.e. } \lambda \in \mathbb{R}, \quad (3.7)$$

with

$$\|\zeta\|_{L^1(\mathbb{R})} = \operatorname{tr}(V).$$

Proof. Let $V = \sum_{k=1}^{\infty} \alpha_k P_k$ be the spectral decomposition of the trace class operator V , where P_k , $k = 1, 2, \dots$, are one-dimensional spectral projections and $\alpha_1 \geq \alpha_2 \geq \dots$, are the corresponding eigenvalues counting multiplicity. Introducing the accumulative operators

$$H_{k+1} = H_k - i\alpha_k P_k, \quad k \in \mathbb{N},$$

and taking into account the multiplicativity of the perturbation determinant [32, Theorem 3.5], one obtains

$$\det_{H_n/H_0}(z) = \prod_{k=1}^n \det_{H_k/H_{k-1}}(z), \quad z \in \mathbb{C}_+. \quad (3.8)$$

By Lemma 3.1, every factor in the product (3.8) is an outer function in \mathbb{C}_+ , so is $\det_{H_n/H_0}(z)$, and, moreover, one has the representations

$$\det_{H_n/H_0}(z) = \exp \left(\frac{1}{\pi i} \int \frac{\zeta_n(\lambda)}{\lambda - z} d\lambda \right), \quad z \in \mathbb{C}_+, \quad n \in \mathbb{N},$$

where $\{\zeta_k\}_{k \in \mathbb{N}}$ is a monotone sequence of nonnegative summable functions. It also follows from Lemma 3.1 that

$$\int_{\mathbb{R}} \zeta_n(\lambda) d\lambda = \sum_{k=1}^n \alpha_k, \quad n \in \mathbb{N}.$$

Since by hypothesis V is a trace class operator, the series $\sum_{k=1}^{\infty} \alpha_k$ converges, and therefore, the sequence ζ_n converges pointwise a.e. and in the topology of the space $L^1(\mathbb{R})$ to a summable function ζ . By [32, Theorem 3.4],

$$\lim_{n \rightarrow \infty} \det_{H_n/H_0}(z) = \det_{H/H_0}(z)$$

uniformly on compact subsets of \mathbb{C}_+ and

$$\lim_{n \rightarrow \infty} \exp \left(\frac{1}{\pi i} \int \frac{\zeta_n(\lambda)}{\lambda - z} d\lambda \right) = \exp \left(\frac{1}{\pi i} \int \frac{\zeta(\lambda)}{\lambda - z} d\lambda \right).$$

Thus, one obtains the representation

$$\det_{H/H_0}(z) = \exp \left(\frac{1}{\pi i} \int \frac{\zeta(\lambda)}{\lambda - z} d\lambda \right), \quad z \in \mathbb{C}_+.$$

In particular, the perturbation determinant $\det_{H/H_0}(z)$ is an outer function (in \mathbb{C}_+) and (3.7) holds. \square

Remark 3.3. An analog of the representation (3.6) with a measure not known to be absolutely continuous has appeared previously in [19, Theorem 9.1]. Recently, (3.6) was extended in [21, Theorem 6.6] to pairs of maximally accumulative operators H_0 and H with trace class differences by treating separately purely imaginary and purely real perturbations and using multiplicativity of the perturbation determinant. Further generalizations of (3.6) can be found in [21, Theorem 5.7].

To obtain an exponential representation for \det_{H/H_0} in \mathbb{C}_- , we remark that the Schwarz reflection principle, which was valid in the self-adjoint setting, does not hold anymore, and it should be modified by the relation

$$\det_{H/H_0}(\lambda - i0) = \det_{H/H^*}(\lambda - i0) \overline{\det_{H/H_0}(\lambda + i0)}, \quad \text{a.e. } \lambda \in \mathbb{R}. \quad (3.9)$$

Theorem 3.4. *Suppose that $H_0 = H_0^*$, $0 \leq V = V^* \in S^1$, and let $H = H_0 - iV$. Then, the perturbation determinant $\det_{H/H_0}(z)$, $z \in \mathbb{C}_-$, admits the representation*

$$\det_{H/H_0}(z) = e^{i\gamma - iaz} B(z) \exp \left(\frac{1}{\pi i} \int_{\mathbb{R}} \frac{1 + \lambda z}{\lambda - z} d\mu(\lambda) \right)$$

$$\times \exp \left(-\frac{1}{\pi i} \int \frac{\zeta(\lambda)}{\lambda - z} d\lambda \right), \quad (3.10)$$

where $\gamma \in \mathbb{R}$, $a \geq 0$, $B(z)$ is the Blaschke product associated with the eigenvalues of H in \mathbb{C}_- , $0 \leq \mu$ is a finite Borel measure on \mathbb{R} , and ζ is the summable function given by (3.7).

Proof. As a consequence of the multiplication rule, we have the decomposition

$$\begin{aligned} \det_{H/H_0}(z) &= \det_{H/H^*}(z) \cdot \det_{H^*/H_0}(z) \\ &= \det_{H/H^*}(z) \cdot \overline{\det_{H/H_0}(\bar{z})}, \quad z \in \rho(H^*) \cap \rho(H_0). \end{aligned} \quad (3.11)$$

By Theorem 3.2 (in accordance with (3.7)), we have

$$\overline{\det_{H/H_0}(\bar{z})} = \exp \left(\frac{1}{\pi i} \int \frac{\zeta(\lambda)}{\lambda - \bar{z}} d\lambda \right) = \exp \left(-\frac{1}{\pi i} \int \frac{\zeta(\lambda)}{\lambda - z} d\lambda \right), \quad z \in \mathbb{C}_-. \quad (3.12)$$

It was established in [19, eq. (8.16)] that in the lower half-plane \mathbb{C}_- , the perturbation determinant $\det_{H/H^*}(z)$, $z \in \mathbb{C}_-$, is an analytic contractive function. Thus, by the standard inner-outer factorization (see Theorem 2.1 and Remark 2.2),

$$\det_{H/H^*}(z) = e^{i\gamma - iaz} B(z) \exp \left(\frac{1}{\pi i} \int_{\mathbb{R}} \frac{1 + \lambda z}{\lambda - z} d\mu(\lambda) \right), \quad z \in \mathbb{C}_-. \quad (3.13)$$

Combining (3.11)–(3.13) completes the proof. \square

4. THE ARGUMENT OF THE PERTURBATION DETERMINANT

Let $H_0 = H_0^*$ and let $0 \leq V = V^* \in S^1$. Let $\{V_n\}_{n=1}^\infty$ be finite-rank approximations to V with $V_n \geq 0$, $\text{rank}(V_n) \leq n$, and $V_n \rightarrow V$ in the trace class norm. Denote $H = H_0 - iV$ and $H_n = H_0 - iV_n$. Introducing the spectral shift functions $\xi_n(\lambda)$ associated with the pairs H_n and H_0 by the standard relation

$$\xi_n(\lambda) = \frac{1}{\pi} \arg(\det_{H_n/H_0}(\lambda + i0)), \quad \text{a.e. } \lambda \in \mathbb{R},$$

by (3.3) and (3.8), one obtains the bounds

$$-\frac{n}{2} \leq \xi_n(\lambda) \leq \frac{n}{2}, \quad \text{a.e. } \lambda \in \mathbb{R}, \quad n \in \mathbb{N}. \quad (4.1)$$

Thus, in addition to (3.6), one also has the following exponential representation for all $z \in \mathbb{C}_+$,

$$\det_{H/H_0}(z) = |\det_{H/H_0}(i)| \exp \left(\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \xi_n(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\lambda \right). \quad (4.2)$$

However, in general, one cannot bring the limit under the integral due to the fact that the limit of $\xi_n(\lambda)$,

$$\xi(\lambda) = \frac{1}{\pi} \arg(\det_{H/H_0}(\lambda + i0)), \quad \text{a.e. } \lambda \in \mathbb{R}, \quad (4.3)$$

can be non-locally integrable, and hence, not in $L^1(\mathbb{R}, \frac{d\lambda}{1+\lambda^2})$ as discussed in Example 6.4 (see also [22, Ex. 3.10]). We remark that the membership $\xi \in L^1(\mathbb{R}, \frac{d\lambda}{1+\lambda^2})$ can be recovered if the perturbation is slightly stronger than the trace class (see, e.g., [22]). Nonetheless, the spectral shift function $\xi(\lambda)$ given by (4.3) is an element of the larger space of A-integrable functions $(A)L^1(\mathbb{R}, \frac{d\lambda}{1+\lambda^2})$ defined below.

Definition 4.1. Let (X, \mathcal{A}, μ) be a σ -finite measure space, the space $L_{w,0}^1(X, \mu)$ consists of all measurable functions $f(x)$ that satisfy

$$\mu\{x : |f(x)| > t\} = o\left(\frac{1}{t}\right) \quad \text{as } t \rightarrow \infty \text{ and } t \rightarrow 0^+. \quad (4.4)$$

For finite measures μ , the above condition as $t \rightarrow 0^+$ is automatically satisfied. A function f is said to be A -integrable (see, e.g., [6]), if $f \in L_{w,0}^1(X, \mu)$ and the limit

$$(A) \int_X f(x) d\mu(x) := \lim_{\substack{b \rightarrow 0 \\ B \rightarrow \infty}} \int_{\{y \in X : b \leq |f(y)| \leq B\}} f(x) d\mu(x)$$

exists. By definition, the space $(A)L^1(X, \mu)$ consists of all A -integrable functions.

By changing the type of integration in (4.2) we obtain the following exponential representation for the perturbation determinant in terms of the spectral shift function $\xi(\lambda)$.

Theorem 4.2. Let H_0 be a maximal accumulative operator, $0 \leq V = V^* \in S^1$, and define $H = H_0 - iV$. Then

$$\det_{H/H_0}(z) = |\det_{H/H_0}(i)| \exp \left((A) \int_{\mathbb{R}} \frac{1+z\lambda}{\lambda-z} \xi(\lambda) \frac{d\lambda}{1+\lambda^2} \right), \quad z \in \mathbb{C}_+. \quad (4.5)$$

To avoid confusion, we stress that the A -integral on the right hand-side of (4.5) should be understood as the A -integral of the function $\frac{1+z\lambda}{\lambda-z} \xi(\lambda)$, considered as an element of the weighted space $(A)L^1(\mathbb{R}, \frac{d\lambda}{1+\lambda^2})$.

The proof of the theorem is based on the following version of Herglotz-type A -integral formula for functions analytic in \mathbb{C}_+ .

Lemma 4.3. If f is analytic in \mathbb{C}_+ with boundary values in $L_{w,0}^1(\mathbb{R}; \frac{d\lambda}{1+\lambda^2})$, then $\operatorname{Re} f$ and $\operatorname{Im} f$ are A -integrable on \mathbb{R} with respect to $\frac{d\lambda}{1+\lambda^2}$ and

$$f(z) = i \operatorname{Im} f(i) + \frac{1}{\pi i} (A) \int_{\mathbb{R}} \frac{1+\lambda z}{\lambda-z} \operatorname{Re} f(\lambda) \frac{d\lambda}{1+\lambda^2} \quad (4.6)$$

$$= \operatorname{Re} f(i) + \frac{1}{\pi} (A) \int_{\mathbb{R}} \frac{1+\lambda z}{\lambda-z} \operatorname{Im} f(\lambda) \frac{d\lambda}{1+\lambda^2}, \quad z \in \mathbb{C}_+. \quad (4.7)$$

Proof. Let $F(z) = f(i \frac{1-z}{1+z})$. Then $F(z)$ is analytic on the unit disk \mathbb{D} and its boundary value is a function in $L_{w,0}^1(\partial\mathbb{D})$. By Aleksandrov's theorem [9, Theorem 2.3.6], the function F is A -integrable on $\partial\mathbb{D}$ with respect to the Lebesgue measure and

$$F(z) = \frac{1}{2\pi i} (A) \int_{\partial\mathbb{D}} \frac{F(w)}{w-z} dw = (A) \int_0^{2\pi} \frac{F(e^{i\theta})}{1-e^{-i\theta}z} \frac{d\theta}{2\pi}, \quad z \in \mathbb{D}. \quad (4.8)$$

In particular, applying (4.8) to the function $\frac{F(w)}{1-w\bar{z}}$ we get

$$\overline{F(0)} = \overline{(A) \int_0^{2\pi} \frac{F(e^{i\theta})}{1-e^{i\theta}\bar{z}} \frac{d\theta}{2\pi}} = (A) \int_0^{2\pi} \frac{\overline{F(e^{i\theta})}}{1-e^{-i\theta}z} \frac{d\theta}{2\pi}. \quad (4.9)$$

Adding (4.8) and (4.9) yields,

$$F(z) + \overline{F(0)} = (A) \int_0^{2\pi} \frac{2\operatorname{Re} F(e^{i\theta})}{1-e^{-i\theta}z} \frac{d\theta}{2\pi} \quad (4.10)$$

$$\begin{aligned}
&= (A) \int_0^{2\pi} \frac{1 + e^{-i\theta} z}{1 - e^{-i\theta} z} \operatorname{Re} F(e^{i\theta}) \frac{d\theta}{2\pi} + (A) \int_0^{2\pi} \operatorname{Re} F(e^{i\theta}) \frac{d\theta}{2\pi} \\
&= (A) \int_0^{2\pi} \frac{1 + e^{-i\theta} z}{1 - e^{-i\theta} z} \operatorname{Re} F(e^{i\theta}) \frac{d\theta}{2\pi} + \operatorname{Re} F(0), \quad z \in \mathbb{D},
\end{aligned} \tag{4.11}$$

where (4.10) with $z = 0$ was used to evaluate the last integral. Thus,

$$F(z) = i\operatorname{Im} F(0) + (A) \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \operatorname{Re} F(e^{i\theta}) \frac{d\theta}{2\pi}, \quad z \in \mathbb{D}. \tag{4.12}$$

Rewriting (4.12) in terms of $f(z)$ and changing variables under the integral yield (4.6). Replacing $f(z)$ by $if(z)$ in (4.6) gives (4.7). \square

Proof of Theorem 4.2. Let $f(z) = \log(\det_{H/H_0}(z))$, then $\operatorname{Re} f(\lambda + i0) = \zeta(\lambda)$ and $\operatorname{Im} f(\lambda + i0) = \pi\xi(\lambda)$, $\lambda \in \mathbb{R}$. By Theorem 3.2, the function $\zeta(\lambda)$ is in $L^1(\mathbb{R})$ and hence $\frac{1+z\lambda}{\lambda-z} \operatorname{Re} f(\lambda + i0)$ is in $L^1(\mathbb{R}) \subset L^1_{w,0}(\mathbb{R}, \frac{d\lambda}{1+\lambda^2})$, $\operatorname{Im}(z) \neq 0$. Moreover, since the spectral shift function $\xi(\lambda)$ is the Hilbert transform of the $L^1(\mathbb{R})$ function $\zeta(\lambda)$, it follows from [25, (1.6)] that $\xi(\lambda)$ satisfies (4.4) as $t \rightarrow \infty$ and hence so does the function $\frac{1+z\lambda}{\lambda-z} \operatorname{Im} f(\lambda + i0)$, $\operatorname{Im}(z) \neq 0$. Since the measure $\frac{d\lambda}{1+\lambda^2}$ is finite, it follows that $\frac{1+z\lambda}{\lambda-z} \operatorname{Im} f(\lambda + i0)$ is in $L^1_{w,0}(\mathbb{R}, \frac{d\lambda}{1+\lambda^2})$, $\operatorname{Im}(z) \neq 0$. Thus, $f(z)$ satisfies the assumptions of Lemma 4.3 and so (4.5) follows from (4.7). \square

5. A TRACE FORMULA

We will now derive a trace formula for rational functions that may have poles in both \mathbb{C}_+ and \mathbb{C}_- . Denote

$$\mathcal{F} = \operatorname{span}\{\lambda \mapsto (\lambda - z)^{-k} : k \in \mathbb{N}, z \in \rho(H_0) \cap \rho(H) \cap (\mathbb{C} \setminus \mathbb{R})\}.$$

Let \mathcal{P}_\pm be the orthogonal projections onto the Hardy spaces $H^2_\pm(\mathbb{R})$ [17, Chapter VI].

Theorem 5.1. *Suppose that $H_0 = H_0^*$, $0 \leq V = V^* \in S^1$, and let $H = H_0 - iV$. Then,*

$$\begin{aligned}
\operatorname{tr}(f(H) - f(H_0)) &= \sum_k ((\mathcal{P}_+ f)(z_k) - (\mathcal{P}_+ f)(\overline{z_k})) + (A) \int_{\mathbb{R}} f'(\lambda) \xi(\lambda) d\lambda \\
&\quad + \frac{1}{\pi i} \int_{\mathbb{R}} (\mathcal{P}_+ f')(\lambda) (1 + \lambda^2) d\mu(\lambda) - ia \operatorname{Res}|_{w=\infty} (\mathcal{P}_+ f)(w),
\end{aligned} \tag{5.1}$$

for $f \in \mathcal{F}$, where a , μ are as in (3.10) and z_k are eigenvalues of H .

Proof. By the known argument (see, e.g., [13, Chapter IV, § 3.2, Prop. 5]),

$$\operatorname{tr}((H - z)^{-1} - (H_0 - z)^{-1}) = -\frac{\frac{d}{dz} \det_{H/H_0}(z)}{\det_{H/H_0}(z)}, \quad z \in \rho(H) \cap \rho(H_0). \tag{5.2}$$

Therefore, by Theorem 3.2,

$$\operatorname{tr}((H - z)^{-1} - (H_0 - z)^{-1}) = -\frac{1}{\pi i} \int_{\mathbb{R}} \frac{\zeta(\lambda)}{(\lambda - z)^2} d\lambda, \quad z \in \mathbb{C}_+. \tag{5.3}$$

By the representation (5.2) and Theorem 3.4,

$$\operatorname{tr}((H - z)^{-1} - (H_0 - z)^{-1}) = -\sum_k \left(\frac{1}{z - z_k} - \frac{1}{z - \overline{z_k}} \right)$$

$$+ \frac{1}{\pi i} \int_{\mathbb{R}} \frac{\zeta(\lambda)}{(\lambda - z)^2} d\lambda - \frac{1}{\pi i} \int_{\mathbb{R}} \frac{1 + \lambda^2}{(\lambda - z)^2} d\mu(\lambda) + ia, \quad z \in \rho(H) \cap \mathbb{C}_-. \quad (5.4)$$

Next we combine (5.3) and (5.4) to obtain (5.1). Denote

$$f_{z,p}(\lambda) = (\lambda - z)^{-p}, \quad p \in \mathbb{N}, \quad z \in \rho(H_0) \cap \rho(H) \cap (\mathbb{C} \setminus \mathbb{R}). \quad (5.5)$$

The series $\sum_k \left(\frac{1}{z - z_k} - \frac{1}{z - \bar{z}_k} \right)$ converges uniformly on every compact subset of $\rho(H_0) \cap \rho(H) \cap (\mathbb{C} \setminus \mathbb{R})$. By differentiating (5.3) and (5.4) with respect to z , we obtain

$$\begin{aligned} \operatorname{tr} (f_{z,p}(H) - f_{z,p}(H_0)) &= \frac{1}{\pi i} \int_{\mathbb{R}} f'_{z,p}(\lambda) \zeta(\lambda) d\lambda \times \begin{cases} 1, & z \in \mathbb{C}_+ \\ -1, & z \in \mathbb{C}_- \end{cases} \\ &+ \left(ia + \frac{1}{\pi i} \int_{\mathbb{R}} f'_{z,p}(\lambda) (1 + \lambda^2) d\mu(\lambda) + \sum_k (f_{z,p}(z_k) - f_{z,p}(\bar{z}_k)) \right) \\ &\times \begin{cases} 0, & z \in \mathbb{C}_+ \\ 1, & z \in \mathbb{C}_- \end{cases} \end{aligned} \quad (5.6)$$

for every $p \in \mathbb{N}$, $z \in \rho(H_0) \cap \rho(H) \cap (\mathbb{C} \setminus \mathbb{R})$, where a, μ are as in (3.10).

Denote by \mathcal{T} the Hilbert transform on $L^2(\mathbb{R})$, respectively, so that $\mathcal{T} = \frac{1}{i}(\mathcal{P}_+ - \mathcal{P}_-)$. Fix $z \in \rho(H_0) \cap \rho(H) \cap (\mathbb{C} \setminus \mathbb{R})$ and $p \in \mathbb{N}$. It is easy to see that for $f_{z,p}$ given by (5.5),

$$\mathcal{P}_+ f'_{z,p} = f'_{z,p} \times \begin{cases} 0, & z \in \mathbb{C}_+ \\ 1, & z \in \mathbb{C}_- \end{cases}$$

and that

$$\mathcal{T} f'_{z,p} = f'_{z,p} \times \begin{cases} -1, & z \in \mathbb{C}_+ \\ 1, & z \in \mathbb{C}_- \end{cases}.$$

Hence, the trace formula (5.6) can be rewritten via the Hilbert transform:

$$\begin{aligned} \operatorname{tr} (f_{z,p}(H) - f_{z,p}(H_0)) &= \sum_k ((\mathcal{P}_+ f_{z,p})(z_k) - (\mathcal{P}_+ f_{z,p})(\bar{z}_k)) \\ &- \frac{1}{\pi} \int_{\mathbb{R}} (\mathcal{T} f'_{z,p})(\lambda) \zeta(\lambda) d\lambda + \frac{1}{\pi i} \int_{\mathbb{R}} (\mathcal{P}_+ f'_{z,p})(\lambda) (1 + \lambda^2) d\mu(\lambda) \\ &- ia \operatorname{Res}_{|w=\infty} (\mathcal{P}_+ f_{z,p}(w)). \end{aligned} \quad (5.7)$$

It is proved in [7] that if $\phi \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $p \geq 1$, with $\mathcal{T}(\phi) \in L^\infty(\mathbb{R})$, and $h \in L^1(\mathbb{R})$,

$$\int_{\mathbb{R}} h(x) (\mathcal{T} \phi)(x) dx = -(A) \int_{\mathbb{R}} (\mathcal{T} h)(x) \phi(x) dx$$

(see also [37] for the analogous result on the unit circle), which along with (5.7) gives us (5.1) for $f = f_{z,p}$. Taking linear combinations of functions $f_{z,p}$ extends (5.1) to all $f \in \mathcal{F}$. \square

Remark 5.2. (i) If H and H_0 is a pair of self-adjoint operators with $H - H_0 \in S^1$, then the analog of (5.1) has a simpler form:

$$\operatorname{tr} (f(H) - f(H_0)) = \int_{\mathbb{R}} f'(\lambda) \xi(\lambda) d\lambda, \quad (5.8)$$

as established in [18]. A detailed list of references on (5.8) can be found in the surveys [8, 33]; references on higher order trace formulas can be found in [33]. Attempts to extend the trace

formula (5.8) to accumulative operators H_0 and H resulted in consideration of only selected pairs of accumulative H_0 and H and led to modification of either the left or right hand side of (5.8) [4, 5, 19, 22, 23, 26, 31]. It is also known [10] that for every pair of maximal accumulative operators H_0 and H , with $H - H_0 \in S^1$, there exists a finite measure μ such that $\|\mu\| \leq \|V\|_1$ and

$$\begin{aligned} \operatorname{tr} (f(H) - f(H_0)) &= \int_{\mathbb{R}} f'(\lambda) d\mu(\lambda), \\ f &\in \operatorname{span}\{\lambda \mapsto (z - \lambda)^{-k} : k \in \mathbb{N}, z \in \mathbb{C}_+\}. \end{aligned}$$

(ii) By adjusting the reasoning in the proof of Theorem 5.1 to the perturbation determinant $\det_{H/H^*}(z)$, we obtain that for $H_0 = H_0^*$, $0 \leq V = V^* \in S^1$, and $H = H_0 - iV$,

$$\begin{aligned} \operatorname{tr}(f(H) - f(H^*)) &= \sum_k (f(z_k) - f(\bar{z}_k)) \\ &+ \frac{1}{\pi i} \int_{\mathbb{R}} f'(\lambda)(1 + \lambda^2) d\mu(\lambda) - ia \operatorname{Res}_{|w=\infty} (f(w)), \end{aligned} \quad (5.9)$$

for rational functions $f \in C_0(\mathbb{R})$ with poles in $\rho(H) \cap \rho(H^*) \cap \mathbb{C}_-$, where a, μ are as in (3.10). By taking complex conjugation in (5.9), we extend the formula to all rational functions $f \in C_0(\mathbb{R})$ with poles in $\rho(H) \cap \rho(H^*)$. The formula (5.9) was obtained in [5, Theorem 1] using a functional model of accumulative operators and in [19, Theorem 8.3 and 8.4] via the perturbation determinant. A similar formula for bounded dissipative operators with absolutely continuous spectrum was obtained earlier in [31].

(iii) Under the assumptions of Theorem 5.1, we also have the trace formula

$$\operatorname{tr} (f(H) - f(H_0)) = \frac{1}{\pi i} \int_{\mathbb{R}} f'(\lambda) \zeta(\lambda) d\lambda,$$

where f is a rational function with poles in \mathbb{C}_+ . This follows from the formula (5.3), which also appeared in [21, Theorem 6.6].

(iv) Since the functions $\pi\xi(\lambda)$ and $\zeta(\lambda)$ are harmonic conjugates of each other, one can avoid appearance of the A -integral in the trace formula (5.1) using the equality

$$(A) \int_{\mathbb{R}} f'(\lambda) \xi(\lambda) d\lambda = -\frac{1}{\pi} \int_{\mathbb{R}} (\mathcal{T}f')(\lambda) \zeta(\lambda) d\lambda,$$

with \mathcal{T} the Hilbert transform and standard Lebesgue integral on the right-hand side.

(v) The trace formula (5.1) is an accumulative analog of a regularized trace formula obtained by A. Rybkin in [29] for contractive trace class perturbations of a unitary operator. However, it is worth mentioning that Rybkin's approach requires a concept of a spectral shift distribution and invokes B-integration in the corresponding trace formula.

6. NON-INTEGRABILITY OF THE SPECTRAL SHIFT FUNCTION

In this concluding section we discuss two important examples that emphasize some properties of the spectral shift function that are not available in the standard trace class perturbation theory for self-adjoint operators.

We start with the observation that since ξ is the Hilbert transform of an integrable function, one automatically has that $\xi \in L_w^1(\mathbb{R})$, the weak L^1 space. However, the following example shows that $\xi \notin L_{w,0}^1(\mathbb{R}) \subset L_w^1(\mathbb{R})$.

Example 6.1. (cf. [23, Ex. 3.6]) Let $H_0 = 0$ and $H = -\alpha i P$, where $\alpha > 0$ and P is a rank one orthogonal projection. The function ξ for the pair H and H_0 can be computed explicitly

$$\xi(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0^+} \operatorname{Im} (\log(1 + i\alpha(\lambda + i\varepsilon)^{-1})) = \frac{1}{\pi} \arctan \left(\frac{\alpha}{\lambda} \right), \quad (6.1)$$

and hence, $\xi \notin L^1_{w,0}(\mathbb{R})$ since $\arctan(a/\lambda) \sim a/\lambda$ as $\lambda \rightarrow \infty$. Note that the function ζ from Lemma 3.1 is given by

$$\zeta(\lambda) = \lim_{\varepsilon \rightarrow 0^+} \operatorname{Re} (\log(1 + i\alpha(\lambda + i\varepsilon)^{-1})) = \log \sqrt{1 + \frac{\alpha^2}{\lambda^2}}, \quad (6.2)$$

and, therefore, $\zeta \in L^1(\mathbb{R})$.

In fact, the phenomenon of $\xi \notin L^1_{w,0}(\mathbb{R})$ observed in Example 6.1 is of general character. As Theorem 6.2 below shows, the spectral shift function $\xi(\lambda) = \xi(\lambda, H_0, H)$, being the Hilbert transform of a nonnegative integrable function $\zeta(\lambda)$, is never an element of $L^1_{w,0}(\mathbb{R})$, unless the operator H is also self-adjoint. A weaker statement that in the context of an accumulative perturbation the spectral shift function ξ is necessarily not in $L^1(\mathbb{R})$, follows from the claim in [34, 6.1, p. 48] that the Hilbert transform of a positive $L^1(\mathbb{R})$ function is not in $L^1(\mathbb{R})$.

Theorem 6.2. *If $f \in L^1(\mathbb{R})$ is such that $\int_{\mathbb{R}} f(y) dy \neq 0$, then the Hilbert transform of f ,*

$$g(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{y - x} dy, \quad (6.3)$$

is not in $L^1_{w,0}(\mathbb{R})$, and in particular, not integrable.

Proof. For any $h \in L^1(\mathbb{R})$ it follows from the Dominated Convergence Theorem that

$$\begin{aligned} \limsup_{t \rightarrow 0^+} t |\{x : |h(x)| > t\}| &= \limsup_{t \rightarrow 0^+} \int_{\{x : |h(x)| > t\}} t dx \\ &\leq \limsup_{t \rightarrow 0^+} \int_{\mathbb{R}} \min\{t, |h(x)|\} dx = 0. \end{aligned} \quad (6.4)$$

Thus, it suffices to show that g does not satisfy (6.4). In fact, we will show that

$$\limsup_{t \rightarrow 0^+} t |\{x : |g(x)| > t\}| \geq 2 \left| \int_{\mathbb{R}} f(y) dy \right| > 0. \quad (6.5)$$

In the following, we split g into three parts

$$\begin{aligned} g(x) &= \frac{-1}{x} \int_{-M}^M f(y) dy + \text{p.v.} \int_{-M}^M \frac{yf(y)}{x(y-x)} dy + \text{p.v.} \int_{|y| > M} \frac{f(y)}{y-x} dy \\ &=: g_{0,M}(x) + g_{1,M}(x) + g_{2,M}(x). \end{aligned} \quad (6.6)$$

For any $0 < \varepsilon < 1/2$ and $t > 0$, the inequality $|g_{0,M}(x)| = |g(x) - g_{1,M}(x) - g_{2,M}(x)| > t$ implies that either $|g(x)| > (1 - 2\varepsilon)t$ or $|g_{1,M}(x)| > \varepsilon t$ or else $|g_{2,M}(x)| > \varepsilon t$. Hence,

$$\begin{aligned} |\{x : |g(x)| > (1 - 2\varepsilon)t\}| &\geq |\{x : |g_{0,M}(x)| > t\}| \\ &\quad - |\{x : |g_{1,M}(x)| > \varepsilon t\}| - |\{x : |g_{2,M}(x)| > \varepsilon t\}|. \end{aligned} \quad (6.7)$$

Since the function $g_{0,M}(x)$ is a constant multiple of $1/x$, we compute

$$\lim_{M \rightarrow \infty} \limsup_{t \rightarrow 0^+} t |\{x : |g_{0,M}(x)| > t\}| = \lim_{M \rightarrow \infty} 2 \left| \int_{-M}^M f(y) dy \right| = 2 \left| \int_{\mathbb{R}} f(y) dy \right|. \quad (6.8)$$

Using the inequality $\left| \int_{-M}^M \frac{yf(y)}{x(y-x)} dy \right| \leq \frac{2M\|f\|_1}{|x|^2}$ for all $|x| > 2M$, we estimate

$$\limsup_{t \rightarrow 0^+} t |\{x : |g_{1,M}(x)| > t\}| \leq \limsup_{t \rightarrow 0^+} 2t \left(2M + \sqrt{\frac{2M\|f\|_1}{t}} \right) = 0. \quad (6.9)$$

Denoting by C the norm of the Hilbert transform as a map from $L^1(\mathbb{R})$ to $L_w^1(\mathbb{R})$, we obtain

$$\lim_{M \rightarrow \infty} \sup_{t > 0} t |\{x : |g_{2,M}(x)| > t\}| \leq \lim_{M \rightarrow \infty} C \int_{|y| > M} |f(y)| dy = 0. \quad (6.10)$$

Finally, combining the above estimates (6.7)–(6.10) implies

$$\begin{aligned} \limsup_{t \rightarrow 0^+} t |\{x : |g(x)| > t\}| &= \limsup_{t \rightarrow 0^+} (1 - 2\varepsilon) t |\{x : |g(x)| > (1 - 2\varepsilon)t\}| \\ &\geq (1 - 2\varepsilon) \lim_{M \rightarrow \infty} \limsup_{t \rightarrow 0^+} \left[t |\{x : |g_{0,M}(x)| > t\}| \right. \\ &\quad \left. - t |\{x : |g_{1,M}(x)| > \varepsilon t\}| - t |\{x : |g_{2,M}(x)| > \varepsilon t\}| \right] \\ &\geq (1 - 2\varepsilon) 2 \left| \int_{\mathbb{R}} f(y) dy \right|. \end{aligned} \quad (6.11)$$

Since g does not satisfy (6.4), it is not in $L_{w,0}^1(\mathbb{R})$ and hence not in $L^1(\mathbb{R})$. \square

Remark 6.3. It follows from the proofs of Theorem 4.2 and Theorem 6.2 that as long as the perturbation V is not zero, the spectral shift function $\xi(\lambda)$ satisfies

$$\limsup_{t \rightarrow \infty} t |\{\lambda : |\xi(\lambda)| > t\}| = 0 \quad \text{and} \quad \limsup_{t \rightarrow 0^+} t |\{\lambda : |\xi(\lambda)| > t\}| > 0. \quad (6.12)$$

Our second example shows that the spectral shift function does not even need to be locally integrable.

Example 6.4. (cf. [23, Ex. 3.10]) Let $H_0 = 0$ and $H = -i \sum_{n=1}^{\infty} \alpha_n P_n$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a summable sequence of positive numbers so that $\sum_{n=1}^{\infty} \alpha_n \ln(\alpha_n)$ is divergent and $\{P_n\}_{n=1}^{\infty}$ is a sequence of rank one orthogonal projections such that $P_n P_k = 0$ whenever $n \neq k$. As in the previous example, the functions ξ and ζ for the pair H and H_0 can be computed explicitly

$$\xi(\lambda) = \frac{1}{\pi} \sum_{n=1}^{\infty} \arctan\left(\frac{\alpha_n}{\lambda}\right) \quad \text{and} \quad \zeta(\lambda) = \sum_{n=1}^{\infty} \log \sqrt{1 + \frac{\alpha_n^2}{\lambda^2}}. \quad (6.13)$$

Since $\int_0^1 \arctan(\frac{\alpha_n}{\lambda}) d\lambda = \frac{\alpha_n}{2} \ln(1 + \alpha_n^2) + \arctan(\alpha_n) - \alpha_n \ln(\alpha_n)$, it follows from the monotone convergence theorem and the divergence of $\sum_{n=1}^{\infty} \alpha_n \ln(\alpha_n)$ that $\int_0^1 |\xi(\lambda)| d\lambda = \infty$. Hence, ξ is not locally integrable and, in particular, not in $L^1(\mathbb{R}; \frac{d\lambda}{1+\lambda^2})$. On the other hand, since $\int_{\mathbb{R}} \log \sqrt{1 + \frac{\alpha_n^2}{\lambda^2}} d\lambda = \alpha_n \int_{\mathbb{R}} \log \sqrt{1 + \frac{1}{\lambda^2}} d\lambda$, it follows from the monotone convergence theorem and the summability assumption on $\{\alpha_n\}_{n=1}^{\infty}$ that $\zeta \in L^1(\mathbb{R})$.

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